The V-flex, Triangle Orientation, and Catalan Numbers in Hexaflexagons

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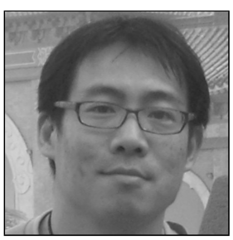
T. Bruce McLean (bmclean@georgiasouthern.edu) was introduced to flexagons at a 1963 MAA meeting in Columbus, Ohio by Ohio Northern University Professor Earl Lhamon. While Bruce taught at Findlay High School and Ohio Northern, students Bob Verrey and Alan Moluf helped him expand knowledge of the V-flex. He received his M.A. from Bowling Green State University in 1965 when he submitted his first paper on the V-flex. He received his Ph.D. from the University of Kentucky under Brauch Fugate in 1971. His area of research is one dimensional continua.



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Using only the pinch flex, the flex described by Martin Gardner in [**4**], the six triangles of each hexagonal face of a hexaflexagon stay together. If the faces are colored, the face facing up is always monochrome. To scramble the triangles and mix the colors, you need other flexes. In this paper, we describe the V-flex. With the V-flex, faces become multi-colored when flexed. It takes persistence to master, but the V-flex is worth it. A hexahexaflexagon has only 9 mathematical faces under the pinch flex; with the V-flex it has 3,420.

We conjecture that many people performed a V-flex accidentally as they read Martin Gardner's 1956 article and had no expert help. In 1963, Bruce McLean discovered the V-flex and, in 1978, provided a graph of the faces. He presented it to Arthur Stone at

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an MAA meeting that year and was told "I never allowed my students to do that!" In that paper, McLean approached the V-flex systematically as a new flex. The view that the V-flex is somehow *wrong*, however, persists to the present. In 2011, a referee of this paper wrote, "I've taken the majority view, that it was an illegal move and that my only purpose in understanding it was to fix 'broken' flexagons." The V-flex is now understood as a well-defined operation (see [**8**] for an overview) that adds considerable complexity to an already interesting object. After the meeting with McLean, Stone expressed pleasure that order still prevailed under the V-flex. We hope the reader will be pleased too.

V-flexing Begin by constructing a hexahexaflexagon (see Figure 2A in [**4**]). For clarity, we prefer to number all triangles in the strip of paper from left to right on the front and back of the paper as in our Figure 1, and add arrows pointing to each triangle edge *not* shared with another triangle. The *tab* is helpful for construction, but not required.

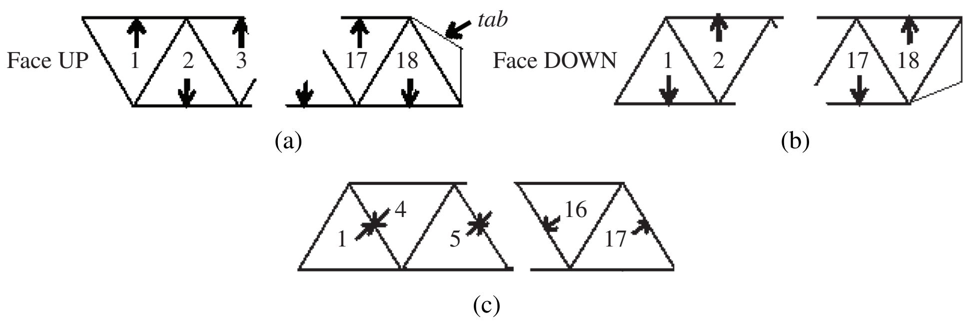

Figure 1. Constructing a hexahexaflexagon

Fold the triangle UP 2 under the 1 and continue to spiral in the same direction to obtain a shorter strip of 9 triangles as in Figure 1c. Next, fold triangle UP 5 under the 4, UP 11 under the 10, and UP 17 under the 16. Finish by taping the tab to UP 1. The result is the hexahexaflexagon shown in Figure 2a. Note that each triangular region of the hexahexaflexagon, called a *pat*, has a thickness, i.e., the number of triangles. This number is the *degree* of the pat. The strip would continue straight and a hexagon could not be constructed if the degree of any pat was divisible by three, hence pat degrees are always congruent to 1 or 2 (mod 3).

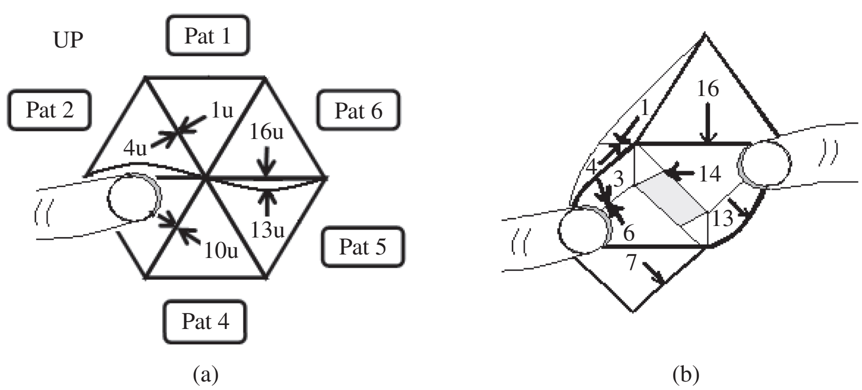

Figure 2. Starting the V-flex.

For the V-flex, separate Pat 2 with your left thumb into two halves (see Figure 2a). It may help to fold the whole flexagon in half, so that 7, 10 and 13 are on one side. You may also need your right thumb to assist in separating Pat 5. The flexagon should begin to open (Figure 2b) and eventually look like Figure 3a (as seen from the top). Note that Figure 3a is *three-dimensional*: an inverted pyramid with a square base and tetrahedra on two opposite faces. Pinch 13 with your right thumb on top and pull it towards you while pushing 6 away from you. Rotating 13 under 11 completes the flex (see Figure 3b). (An online video may be helpful; see [**9**].)

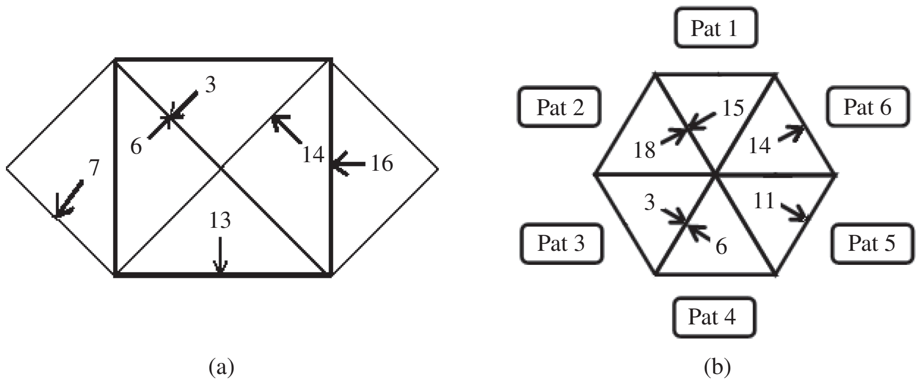

Figure 3. Completing the V-flex.

Note that a V-flex is *not* possible if either Pat 2, 5, or 6 contains only 1 triangle. To return to the original, flip the flexagon on the horizontal axis, rotate 60 degrees clockwise, and V-flex again. Enjoy!

Triangle orientation Many flexagon lovers wrote Martin Gardner [**5**] to point out that numbers alone are not enough to describe a flexagon configuration completely. A triangle can come up face up or face down *and* with different orientations, that is, the arrow mentioned in the previous section may point in different directions within the face.

We define the *state* of a triangle as the following three pieces of information: (i) face position: up (U) or down (D); (ii) triangle number: even (E) or odd (O); and (iii) triangle orientation (the arrow's direction): outside the hexagon (O), towards the next pat in the face counterclockwise (N), or towards the previous pat in the face (P). There are thus 12 distinct states. For instance, the state UOO means that the triangle is facing up, is odd-numbered, and the arrow points outwards in the hexaflexagon; the state DOP means a face down triangle, odd-numbered, and with the arrow pointing clockwise.

Simple observations lead to the fact that from one triangle to the next, in increasing numerical order, parity changes (evidently), but facing up or down changes only if both triangles are in the same pat. We say that a transition from one triangle to the next one is a *folding transition* (FT), if both triangles are in the same pat; or a *pat transition* (PT), if they are in consecutive pats. Then, it is easy to observe that, for instance, from a DON triangle we can only have a folding transition (FT) to the next triangle in the sequence, which therefore must be UEO (U because it is in the same pat; E because it comes after an odd numbered triangle; and O as one can determine by looking at the original strip of paper).

We summarize these observations with the labeled, directed graph in Figure 4, where each edge is either a FT (folding transition) or a PT (pat transition). The diagram shows clearly which states allow a FT or PT, or both, and that the number of

Q: No figure 5 provided. Changed to Figure 4 here and in caption.

triangles folded in consecutive pats forms a sequence consisting of numbers of leaves congruent alternately to 1 and 2 (mod 3).

Given any snapshot of triangle numbers in a pat, we can easily determine the complete state (with orientation) of *every* triangle. For instance, consider the pat consisting of triangles 2, 1, 4, 3 (from top to bottom) in a hexahexaflexagon. Triangle 1 must be at a tip of a PT edge (hence in either DOO or UON states) and must begin a path of three FT edges leading to a state that allows the next PT (since the pat contains 4 triangles). This puts triangle 1 in DOO state. Then, as we move to triangle 2, we must follow a FT edge (it is the same pat), which places triangle 2 in UEP state. We continue with a FT to 3 in DON, and another FT to 4 in UEO.

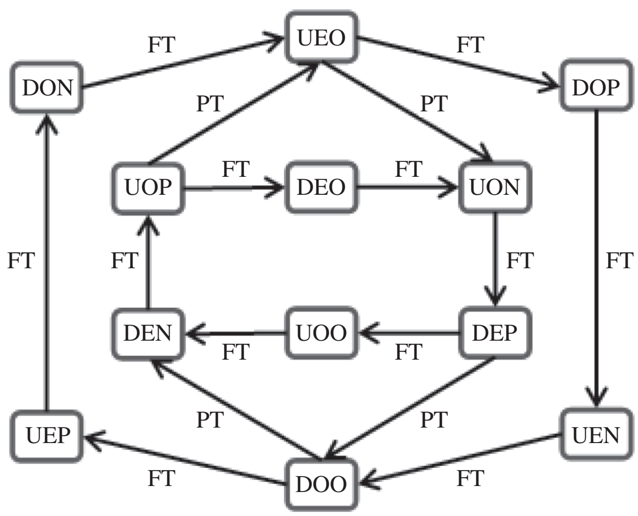

Figure 4.

Catalan numbers in flexagons The enumeration of pat classes (under a natural equivalence relation according to how the degree splits, see [**1**]) enjoys a beautiful connection to the well-known Catalan numbers, which we proceed to describe. A pat of degree n, $n > 1$, splits naturally where a “thumbhole” is formed, the one place in the physical pat where the thumb can be inserted without encountering a pocket [**10**]. This divides the pat into two parts whose degrees sum to n. Thus each pat class can be represented as a partition of n into an ordered sum of two positive integers representing the degrees of its two principle parts. Like n, these integers are equivalent either to 1 or to 2 (mod 3). Then each part in this partition is in turn a sum of two such parts and so on, until every part equals 1. For example, a pat class of degree 10 may be represented as follows:

$$\begin{aligned}
10 &= 2 + 8\\
&= (1+1) + (1+7)\\
&= (1+1) + (1+(5+2))\\
&= (1+1) + (1+((1+4)+(1+1)))\\
&= (1+1) + (1+((1+(2+2))+(1+1)))\\
&= (1+1) + (1+((1+((1+1)+(1+1)))+(1+1))).
\end{aligned}$$

From a combinatorial point of view, such a representation assigns pairs of parenthesis to the sum of n 1's keeping the sum within each pair of parenthesis either 1 or 2 modulo 3. The number of pat classes is the number of ways to complete this process. The well known Catalan numbers (which Martin Gardner also popularized in [**3**] count, among other objects (including planar trees, lattice paths, triangulations of a polygon), exactly this quantity without the (mod 3) constraint. The number of pat classes can therefore be analyzed using the same combinatorial tools used for the Catalan numbers (see [**2**]). In the recent paper of Anderson et al. [**1**], a pair of recursive relations for the number of pat classes with a given degree is given, namely

$$b_n = \sum_{i=0}^{n} a_i a_{n-i} \quad \text{and} \quad a_n = \sum_{i=0}^{n-1} b_i b_{n-i-1},$$

where a_n is the number of pat classes of degree $3n+1$ and b_n is the number of pat classes of degree $3n+2$. The reader may have already noticed that when partitioning a number of the form $3n+1$, the two parts must be of the form $3n+2$, and vice versa. This is the reason for this bi-variable recursion.

The enumeration of various objects related to the flexagons that Martin Gardner introduced to a wide audience in 1952 is now an interesting area of research. Both sequences $\{a_n\}$ and $\{b_n\}$ are in the On-Line Encyclopedia [**11**]. Among other interesting concepts associated with a_n are the number of 4-ary (or quartic) trees with $3n+1$ leaves. It turns out that

$$a_n = \frac{1}{3n+1}\binom{4n}{n} \quad \text{and} \quad b_n = \frac{2}{3n+2}\binom{4n+1}{n}.$$

(For more on the combinatorics of flexagons, see [**1**] and [**6**].)

Summary. Regular hexaflexagons mysteriously change faces as you pinch flex them. This paper describes a different flex, the V-flex, which allows the hexahexaflexagon (with only 9 faces under the pinch flex) to have 3420 faces. The article goes on to explain the classification of triangle orientations in a hexaflexagon and gives an example of the combinatorics of flexagons.

References

1. T. Anderson, T. B. McLean, H. Pajoohesh, C. Smith, The combinatorics of all regular flexagons, *European Journal of Combinatorics* **31** (2010) 72–80; available at http://dx.doi.org/10.1016/j.ejc.2009.01.005
2. R. Brualdi, *Introductory Combinatorics*, 5th ed., Pearson, 2008.
3. M. Gardner, Catalan Numbers, *Time Travel and Other Mathematical Bewilderments*, W. H. Freeman, 1987.
4. ———, Hexaflexagons, *College Math. J.*, **43** (2011) 2–5; available at http://dx.doi.org/10.4169/college.math.j.43.1.002
5. ———, *Hexaflexagons and Other Mathematical Diversions*, Cambridge U. Press, 2008.
6. I. E. Iacob, T. B. McLean, and H. Wang, About general order regular flexagons (submitted).
7. T. B. McLean, V-Flexing the hexahexaflexagon, *Amer. Math. Monthly* **86** (1979) 457–466; available at http://dx.doi.org/10.2307/2320415
8. T. B. McLean, Flexagons; available at http://math.georgiasouthern.edu/~bmclean/flex/index.html
9. R. Moseley, Flexagon.net; available at http://www.flexagon.net/index.php?option=com_content&task=view&id=22
10. C. O. Oakley and R. J. Wisner, Flexagons, *Amer. Math. Monthly* **64** (1957), 143–154; available at http://dx.doi.org/10.2307/2310544
11. N. Sloane, The On-Line Encyclopedia of Integer Sequences; available at http://oeis.org/